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Approximation and Ergodic Theorems

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1. INTRODUCTION

A method of approximation of an automorphism T on a nonatomic Lebesgue space by sequences $\{T_n\}$ of periodic transformations was developed by Katok and Stepin in [4] and [5] and later investigated and modified by others in [1] and [2]. The main point of this paper is to further develop the relationship between the spectral structure of the approximating transformations T_n and the spectral structure of T employing the mean ergodic theorem as the main tool.

In Section 3, two very useful corollaries to the mean ergodic theorem are developed. The main theorem in that section is a version of the mean ergodic theorem giving strong convergence of certain averages of operators associated with the approximating transformations T_n to projection operators associated with T. This in turn is used to develop strong convergence of projection operators associated with T_n to projection operators of T. A natural set relationship between the sets $\Lambda(T_n)$ of eigenvalues of T_n and the set $\Lambda(T)$ of eigenvalues of T is obtained. This relationship is investigated in Section 4. As an application of this theory, several results in [2, 5, 7] can be strengthened. In [7], results were established for transformations produced by stacking procedures. These stacking procedures represent specific examples of approximation by periodic transformations. Some of these theorems are extended to the more general case here. This in turn adds a characterization of complete ergodicity and weak mixing properties in terms of convergence of projection operators of T_n.

2. DEFINITIONS AND PRELIMINARIES

Let $(X, \mathbb{Q}, \mu)$ be a measure space and let T be an invertible bimeasurable point transformation mapping X onto X. T is *measure preserving* if $\mu(TA) = \mu(A) = \mu(T^{-1}A)$, $A \in \mathbb{Q}$. T is *ergodic* if $\mu(A) > 0$ and $TA \subset A$ implies $\mu(A) = 1$. T is *completely ergodic* if T^k is ergodic for each positive integer k. Each invertible measure preserving transformation T induces on

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$L_2(X)$ a unitary operator U_T defined by $U_T f(x) = f(T(x))$. By the spectral properties of T, we mean the spectral properties of U_T. An ergodic transformation T is *weakly mixing* if 1 is the only eigenvalue of T. T has *discrete spectrum* if the orthogonal sum of all subspaces of eigenfunctions is the whole space $L_2(X)$. In general, strong convergence of a sequence $\{V_n\}$ of operators on a complex Hilbert space H to an operator V on H will be denoted by $V_n \Rightarrow V$; i.e., for each $f \in H$, $V_n f \to Vf$ as $n \to \infty$. In what follows, by a *transformation* T we mean an invertible bimeasurable mapping which maps X onto X and is measure preserving. $(X, \mathbb{Q}, \mu)$ will be a normalized nonatomic Lebesgue space; i.e., a measure space isomorphic to the unit interval.

DEFINITION 2.1. If $\xi = \{C(1), C(2),..., C(h)\}$ is a finite collection of pairwise disjoint measurable sets whose union $\xi' \subset X$, then we say ξ is a *partition*.

In general, ξ' is a proper subset of X. We denote the complement of this set by $R = X - \xi'$. By T_ξ, we mean an arbitrary transformation which transposes the elements of ξ cyclically. That is, $T_\xi C(i) = C(i+1)$ for $i = 1, 2,..., h_{-1}$ and $T_\xi C(h) = C(1)$ and T_ξ acts arbitrarily on R.

DEFINITION 2.2. Given a transformation T on X, we say a partition $\xi = \{C(i): i = 1,..., h\}$ is *T-admissible* if $TC(i) = C(i+1)$ for $i = 1, 2,..., h-1$.

DEFINITION 2.3. A sequence $\{\xi(n)\}$ of partitions is said to *converge to the unit partition*, denoted $\xi(n) \to \varepsilon$, provided for each $A \in \mathbb{Q}$ there exists a sequence $\{A(n)\}$ of sets such that $A(n)$ is a union of sets from $\xi(n)$ and such that $\mu(A(n) \Delta A) \to 0$ as $n \to \infty$.

DEFINITION 2.4. Let T be a transformation and $f(n)$ a monotonic sequence of positive numbers such that $f(n) \to 0$ as $n \to \infty$. *T admits a cyclic approximation by periodic transformations* (cyclic a.p.t.) with speed $f(n)$ if there exists a sequence $\{\xi(n)\}$ of partitions where $\xi(n) = \{C_n(1), C_n(2),..., C_n(h(n))\}$ and a corresponding sequence $\{T_{\xi(n)}\}$ of transformations for which

(i) $\xi(n) \to \varepsilon$,

(ii) $\sum_{j=1}^{h(n)} \mu(TC_n(j) \Delta T_{\xi(n)} C_n(j)) \leqslant f(h(n))$.

If in addition to (i) and (ii), we have

(iii) $\xi(n)$ is T-admissible

then *T admits an approximation by partitions with speed* $f(n)$.

The notion of cyclic a.p.t. can be found in [4] and that of approximation by partitions in [1].

We remark that the term periodic is used here in the context of permutations on the elements of $\xi(n)$ and does not necessarily imply pointwise periodic. Also, it is clear from the definition of $T_{\xi(n)}$ that the epxression $\sum_{j=1}^{h(n)} \mu(TC_n(j)\,\Delta T_{\xi(n)} C_n(j))$ is independent of the particular $T_{\xi(n)}$ chosen and how $T_{\xi(n)}$ is defined on $R(n)$. Actually, Katok and Stepin only assume $T_{\xi(n)}$ preserves the measure of the elements of $\xi(n)$ and so $\mu(C_n(j)) = 1/h(n)$ for $i = 1, 2, \ldots, h(n)$. The additional assumptions on $T_{\xi(n)}$ used here involve no loss of generality.

We state several results obtained by Katok and Stepin in [5].

THEOREM 2.1. *If T admits a cyclic a.p.t. with speed $f(n) = \theta/n$ and $\theta < 4$, then T is ergodic.*

THEOREM 2.2. *If T admits a cyclic a.p.t. with speed $f(n) = \theta/n$ and $\theta < 2$, then T is not strongly mixing.*

THEOREM 2.3. *If T admits a cyclic a.p.t. with respect to $\{\xi(n)\}$ with arbitrary speed $f(n)$, then T has discrete spectrum and all its eigenvalues are roots of unity.*

A sequence $\{a(n)\}$ of numbers is called a (k,j) *pair sequence* if there exists a k and a j with $j \neq 0$ such that $a(2n) = ka(2n-1) + j$ for $n = 1, 2, \ldots$. The above definition can be found in [7] and was motivated by the definition of Chacon [2] of a *k-pair sequence* which is a (k,j) pair with $j = 1$.

DEFINITION 2.5. A transformation T is said to admit a *cyclic a.p.t. in (k,j) pairs* or *an approximation by partitions in (k,j) pairs with speed $f(n)$* if T admits a cyclic a.p.t. or, respectively, an approximation by partitions with speed $f(n)$ and if the sequence $\{h(n)\}$, where $h(n)$ is the number of elements in $\xi(n)$, has a subsequence which is a (k,j) pair sequence.

The following was obtained by Chacon in [2].

THEOREM 2.4. *If T admits a cyclic a.p.t. in k-pairs with speed $f(n) = \theta/n$ and $\theta < 2$, then T is weakly mixing.*

In the proof of the above theorem, the following statement was established.

LEMMA 2.1. *If T admits a cyclic a.p.t. with speed $f(n) = \theta/n$ and $\theta < 2$ and if $\lambda \in \Lambda(T)$, where $\Lambda(T)$ is the set of all eigenvalues of T, then* $\lim_{n\to\infty} \lambda^{h(n)} = 1$.

The next lemma will be used in the following sections. It also indicates that the hypotheses of several following theorems are natural.

LEMMA 2.2. *If T admits an approximation by partitions with some speed, then T admits a cyclic a.p.t. with speed $f(n) = \theta/n$ with $\theta \leqslant 2$.*

Proof. Since T admits an approximation by partitions, there is a sequence $\{\xi(n)\}$ of T-admissible partitions such that $\xi(n) \to \varepsilon$. Let $\gamma = \sup\{h(n)\,\mu(C_n(1) - T_{\xi(n)} C_n(h(n)): n = 1, 2,...\}$ and let $\theta = 2\gamma$. Since $h(n)\,\mu(C_n(1) - T_{\xi(n)} C_n(h(n))) \leqslant h(n)\,\mu(C_n(1)) = \mu(\xi'(n)) \leqslant 1$ for all n, $\gamma \leqslant 1$ and so $\theta \leqslant 2$. Then, since $\xi(n)$ is T-admissible for each n,

$$\begin{aligned}
&\sum_{j=1}^{h(n)} \mu(TC_n(j)\,\Delta T_{\xi(n)} C_n(j)) \\
&\quad= \mu(TC_n(h(n))\,\Delta C_n(1)) \\
&\quad= \mu(TC_n(h(n)) \cup C_n(1)) - \mu(TC_n(h(n)) \cap C_n(1)) \\
&\quad= \mu(TC_n(h(n))) + \mu(C_n(1)) - 2\mu(TC_n(h(n)) \cap C_n(1)) \\
&\quad= 2\mu(C_n(1) - TC_n(h(n))) \\
&\quad= 2h(n)\,\mu(C_n(1) - TC_n(h(n)))/h(n) \\
&\quad\leqslant 2\gamma/h(n) \\
&\quad= \theta/h(n) \\
&\quad= f(h(n))
\end{aligned}$$

where $f(n) = \theta/n$.

3. ERGODIC THEOREMS AND PROJECTIONS

We will adopt the following version of the mean ergodic theorem. A proof of the theorem due to Riesz can be found in [3].

THEOREM 3.1 (THE MEAN ERGODIC THEOREM). *If U is a unitary operator on a complex Hilbert space H and if P^1 is the projection on the space $H^1 = \{g \in H: Ug = g\}$, then $(1/n) \sum_{j=0}^{n-1} U^j \Rightarrow P^1$.*

COROLLARY 3.1.1. *Let K be the unit circle in the complex plane, U a unitary operator on H, and P^λ be the projection on $H^\lambda = \{g \in H: Ug = \lambda g\}$. Then $(1/n) \sum_{j=0}^{n-1} \lambda^{-j} U^j \Rightarrow P^\lambda$ for each $\lambda \in K$.*

COROLLARY 3.1.2. *Let T be a periodic transformation on $(X, \mathbb{Q}, \mu)$ with period p and U its induced unitary operator on $L_2(X)$. Then*

(i) $P^\lambda = (1/p)\sum_{j=0}^{p-1}\lambda^{-j}U^j$ *if* $\lambda^p = 1$ *and* $P^\lambda = 0$ *if* $\lambda^p \neq 1$ and

(ii) *T has discrete spectrum with eigenvalues the pth roots of unity.*

The proofs of the following lemma and theorem were motivated by Riesz's proof of the mean ergodic theorem.

LEMMA 3.1. *For each unitary operator U and each $\lambda \in K$, $H = H^\lambda \oplus Cl(S^\lambda)$, where $Cl(S^\lambda)$ is the closure of S^λ with $S^\lambda = \{f : f = \lambda g - Ug$ for some $g \in H\}$.*

THEOREM 3.2. *If T admits an approximation by partitions with speed θ/n and $\theta < 2$ with respect to $\{\xi(n)\}$, then for each $\lambda \in K$, $(1/h(n))\sum_{j=0}^{h(n)-1}\lambda^{-j}U_n^j \Rightarrow P^\lambda$ with n, where U_n is the unitary operator associated with an arbitrary transformation $T_{\xi(n)}$ and P^λ is the projection on $H^\lambda = \{g \in H : Ug = \lambda g\}$.*

Proof. The result will first be shown for a specific sequence $\{T_{\xi(n)}\}$. For each n, define $T_n = T_{\xi(n)}$ by $T_n(x) = T(x)$ for $x \in \xi'(n) - C_n(h(n))$, $T_n(x) = T^{-h(n)+1}(x)$ for $x \in C_n(h(n))$, and $T_n(x) = x$ for $x \in R(n)$. Note that T_n is periodic with period $h(n)$. For $\lambda \in K$ and each n, define $A_n^\lambda = \sum_{j=0}^{h(n)-1}\lambda^{-j}V_n^j$, where V_n is the unitary operator associated with T_n defined by $V_n f = f(T_n)$.

The following two statements will be established:

$$A_n^\lambda f \to P^\lambda f \quad \text{as} \quad n \to \infty, \qquad \lambda \in K, \quad f \in H^\lambda; \tag{1}$$

$$A_n^\lambda f \to 0 \quad \text{as} \quad n \to \infty, \qquad \lambda \in K, \quad f \in Cl(S^\lambda). \tag{2}$$

Then Lemma 3.1 will be used to establish the conclusion of the theorem for this sequence $\{V_n\}$.

To establish (1), fix $\lambda \in A(T)$ and $f \in H^\lambda$. Upon examining the definition of T_n, we see on the set $C_n(k)$ that $T_n^j = T^j$ for $j = 0, 1, \ldots, h(n) - k$ and $T_n^j = T^{j-h(n)}$ for $j = (h(n) - k) + 1, \ldots, h(n) - 1$. Thus on $C_n(k)$

$$\begin{aligned}
A_n^\lambda f &= (1/h(n)) \sum_{j=0}^{h(n)-1} \lambda^{-j} f(T_n^j) \\
&= (1/h(n)) \sum_{j=0}^{h(n)-k} \lambda^{-j} f(T^j) + (1/h(n)) \sum_{j=h(n)-k+1}^{h(n)-1} \lambda^{-j} f(T^{j-h(n)}) \\
&= (1/h(n)) \sum_{j=0}^{h(n)-k} \lambda^{-j}\lambda^j f + (1/h(n)) \sum_{j=h(n)-k+1}^{h(n)-1} \lambda^{-j}\lambda^{j-h(n)} f \\
&= (1/h(n))(h(n) - k + 1) f + [(k-1)/h(n)]\, \lambda^{-h(n)} f \\
&= f + [(k-1)/h(n)](\lambda^{-h(n)} - 1) f.
\end{aligned}$$

If $\lambda \in K - \Lambda(T)$, $H^\lambda = \{0\}$ and the above equality still holds. We have established

$$A_n^\lambda f = f + [(k-1)/h(n)](\lambda^{-h(n)} - 1)f$$
$$\text{on} \quad C_n(k), \quad 1 \leqslant k \leqslant h(n), \quad \lambda \in K, \quad f \in H^\lambda. \tag{3}$$

Since T is ergodic, if $f \in H^\lambda$, then f has constant absolute value and hence $|f| = M$. Then, using (3) and noting $|A_n^\lambda f| \leqslant M$,

$$\begin{aligned}
\|A_n^\lambda f - P^\lambda f\|^2 &= \|A_n^\lambda f - f\|^2 \\
&= \int_{\xi'(n)} |A_n^\lambda f - f|^2\, d\mu + \int_{R(n)} |A_n^\lambda f - f|^2\, d\mu \\
&= \sum_{j=1}^{h(n)} \int_{C_n(j)} |[(j-1)/h(n)](\lambda^{-h(n)} - 1)f|^2\, d\mu + \int_{R(n)} |A_n^\lambda f - f|^2\, d\mu \\
&\leqslant |\lambda^{-h(n)} - 1|^2 \sum_{j=1}^{h(n)} \int_{C_n(j)} |f|^2\, d\mu + \int_{R(n)} |A_n^\lambda f - f|^2\, d\mu \\
&= |\lambda^{h(n)} - 1|^2 \int_{\xi'(n)} |f|^2\, d\mu + \int_{R(n)} |A_n^\lambda f - f|^2\, d\mu \\
&\leqslant |\lambda^{h(n)} - 1|^2 \mu(\xi'(n))\, M^2 + \mu(R(n))\, 4M^2.
\end{aligned}$$

For $\lambda \in \Lambda(T)$, by Lemma 2.1, $\lambda^{h(n)} \to 1$. Since $\xi(n) \to \varepsilon$, by a little argument, $\mu(R(n)) \to 0$ and so $A_n^\lambda f \to P^\lambda f = f$. If $\lambda \in K - \Lambda(T)$, $H^\lambda = \{0\}$ and the same results follow so (1) is established.

To prove (2), consider the case when $f \in S^\lambda$. Then $f = \lambda g - Ug$ for some $g \in H = L_2(X)$. Thus on $C_n(k)$

$$\begin{aligned}
A_n^\lambda f &= \lambda A_n^\lambda g - A_n^\lambda g(T) \\
&= (1/h(n)) \sum_{j=0}^{h(n)-1} \lambda^{-j+1} g(T_n^j) - (1/h(n)) \sum_{j=0}^{h(n)-1} \lambda^{-j} g(T(T_n^j)) \\
&= (1/h(n)) \sum_{j=0}^{h(n)-k} \lambda^{-j+1} g(T^j) + (1/h(n)) \sum_{j=h(n)-k+1}^{h(n)-1} \lambda^{-j+1} g(T^{j-h(n)}) \\
&\quad - (1/h(n)) \sum_{j=0}^{h(n)-k} \lambda^{-j} g(T^{j+1}) - (1/h(n)) \sum_{j=h(n)-k+1}^{h(n)-1} \lambda^{-j} g(T^{j-h(n)+1}) \\
&= (1/h(n)) \left[\lambda g + \sum_{j=1}^{h(n)-k} \lambda^{-j+1} g(T^j)\right] \\
&\quad + (1/h(n)) \left[\lambda^{-h(n)+k} g(T^{-k+1}) + \sum_{j=h(n)-k+2}^{h(n)-1} \lambda^{-j+1} g(T^{j-h(n)})\right]
\end{aligned}$$

$$- (1/h(n)) \left[\sum_{j=0}^{h(n)-k-1} \lambda^{-j} g(T^{j+1}) + \lambda^{-h(n)+k} g(T^{h(n)-k+1}) \right]$$

$$- (1/h(n)) \left[\sum_{j=h(n)-k+1}^{h(n)-2} \lambda^{-j} g(T^{j-h(n)+1}) + \lambda^{-h(n)+1} g \right]$$

$$= (1/h(n))[(1-\lambda^{-h(n)})\lambda g + \lambda^{-h(n)+k}(g(T^{1-k}) - g(T^{h(n)+1-k}))]. \quad (4)$$

First assume g is bounded, say, $|g(x)| \leqslant M$. So for $f = \lambda g - Ug$ and upon using (4) we have

$$\|A_n f\|^2 = \int_{\xi'(n)} |A_n^\lambda f|^2 \, d\mu + \int_{R(n)} |A_n f|^2 \, d\mu$$

$$= \sum_{k=1}^{h(n)} \int_{C_n(k)} |(1/h(n))[(1-\lambda^{-h(n)})\lambda g + \lambda^{-h(n)+k}(g(T^{1-k})$$

$$- g(T^{h(n)+1-k}))]|^2 \, d\mu$$

$$+ \int_{R(n)} |\lambda A_n^\lambda g - A_n^\lambda g(T)|^2 \, d\mu$$

$$\leqslant \sum_{k=1}^{h(n)} (1/h(n))^2 \, |1-\lambda^{h(n)}|^2 \int_{C_n(k)} |g|^2 \, d\mu$$

$$+ \sum_{k=1}^{h(n)} (1/h(n))^2 \int_{C_n(k)} |g(T^{1-k}) - g(T^{h(n)+1-k})|^2 \, d\mu$$

$$+ 2 \sum_{k=1}^{h(n)} (1/h(n))^2 \, |1-\lambda^{h(n)}| \int_{C_n(k)} |g| \, |g(T^{1-k}) - g(T^{h(n)+1-k})| \, d\mu$$

$$+ \int_{R(n)} |\lambda A_n^\lambda g - A_n^\lambda g(T)|^2 \, d\mu$$

$$\leqslant (1/h(n))^2 \, |1-\lambda^{h(n)}|^2 \mu(\xi'(n)) M^2 + (1/h(n))^2 \mu(\xi'(n)) \, 4M^2$$

$$+ 2(1/h(n))^2 \, |1-\lambda^{h(n)}| \, \mu(\xi'(n)) \, 2M^2 + \mu(R(n)) \, 4M^2.$$

Note that since $\xi(n) \to \varepsilon$, $\mu(R(n)) \to 0$ and $h(n) \to \infty$. Thus

$$A_n^\lambda f \to 0, \qquad f \in S^\lambda \text{ with } f = \lambda g - Ug \text{ and } g \text{ bounded.} \quad (5)$$

We wish to generalize (5) so let $f \in S^\lambda$ with $f = \lambda g - Ug$, where g is not necessarily bounded. Choose a sequence $\{g_k\}$ of bounded functions in H such that $g_k \to g$ and let $f_k = \lambda g_k - Ug_k$. Clearly, $f_k \to f$ as $k \to \infty$, and by (5),

$A_n^\lambda f_k \to 0$ as $n \to \infty$ for each k. Now, since $\|A_n^\lambda\| \leqslant 1$ for all n, we have

$$\begin{aligned}\|A_n^\lambda f\| &= \|A_n^\lambda f - A_n^\lambda f_k + A_n^\lambda f_k\| \\ &\leqslant \|A_n^\lambda(f - f_k)\| + \|A_n^\lambda f_k\| \\ &\leqslant \|f - f_k\| + \|A_n^\lambda f_k\|.\end{aligned}$$

First choose k sufficiently large to make $\|f - f_k\|$ small independently of n, and then choose n sufficiently large to make $\|A_n f_k\|$ small. Hence $A_n^\lambda f \to 0$ for each $f \in S^\lambda$. For $f \in Cl(S^\lambda)$, repeat the argument by approaching f with a sequence $\{f_k\}$ with $f_k \in S^\lambda$ for each k. It follows $A_n^\lambda f_k \to 0$ as $n \to \infty$ for each k and, using the above, (2) is established.

Now, let $f \in H$. By Lemma 3.1, $H = H^\lambda \oplus Cl(S^\lambda)$. Thus there exists an $f_1 \in H^\lambda$ and an $f_2 \in Cl(S^\lambda)$ such that $f = f_1 + f_2$. Then

$$\begin{aligned}\|A_n^\lambda f - P^\lambda f\| &= \|A_n^\lambda(f_1 + f_2) - P^\lambda(f_1 + f_2)\| \\ &= \|A_n^\lambda f_1 + A_n^\lambda f_2 - f_1\| \\ &\leqslant \|A_n^\lambda f_1 - f_1\| + \|A_n^\lambda f_2\|.\end{aligned}$$

By (1), $\|A_n^\lambda f_1 - f_1\| \to 0$ and by (2), $\|A_n^\lambda f_2\| \to 0$. We conclude $A_n^\lambda \Rightarrow P^\lambda$ and the proof of the theorem is complete for this specific sequence of transformations $\{T_n\}$ as previously defined.

Let $\{T_{\xi(n)}\}$ be an arbitrary sequence of transformations defined with respect to $\{\xi(n)\}$ and let $U_n f = f(T_{\xi(n)})$ and $B_n^\lambda = (1/h(n)) \sum_{j=0}^{h(n)-1} \lambda^{-j} U_n^j$ for each n. We must show $B_n^\lambda \Rightarrow P^\lambda$.

Let H_n be the subspace of $L_2(X)$ consisting of all functions which are constant on elements of $\xi(n)$. Let $f \in H$. Since $\xi(n) \to \varepsilon$, it can be shown that there exists a sequence of functions $\{f_n\}$ with $f_n \to f$ and $f_n \in H_n$; namely, f_n is the projection of f on H_n. Then

$$\begin{aligned}\|A_n^\lambda f - B_n^\lambda f\| &= \|A_n^\lambda f - B_n^\lambda f_n + B_n^\lambda f_n - A_n^\lambda f_n + A_n^\lambda f_n - B_n^\lambda f\| \\ &\leqslant \|(A_n^\lambda - B_n^\lambda)(f - f_n)\| + \|A_n^\lambda f_n - B_n^\lambda f_n\| \\ &\leqslant \|A_n^\lambda - B_n^\lambda\| \, \|f - f_n\| + \|A_n^\lambda f_n - B_n^\lambda f_n\| \\ &\leqslant 2\,\|f - f_n\| + \|A_n^\lambda f_n - B_n^\lambda f_n\|. \qquad (6)\end{aligned}$$

For $x \in \xi'(n)$, $T_n^j(x)$ and $T_{\xi(n)}^j(x)$ are in the same element of $\xi(n)$ for $j = 0, 1, \ldots, h(n) - 1$. Since f_n is constant on elements of $\xi(n)$, $f_n(T_n^j(x)) = f_n(T_{\xi(n)}^j(x))$ for $j = 0, 1, \ldots, h(n) - 1$. It follows $A_n^\lambda f_n = B_n^\lambda f_n$ on $\xi'(n)$. First assume f is bounded and say $|f(x)| < M$. Then

$$\|A_n^\lambda f_n - B_n^\lambda f_n\|^2 = \int_{\xi'(n)} |A_n^\lambda f_n - B_n^\lambda f_n|^2 \, d\mu + \int_{R(n)} |A_n^\lambda f_n - B_n^\lambda f_n|^2 \, d\mu$$

$$= \int_{R(n)} |A_n^\lambda f_n - B_n^\lambda f_n|^2 \, d\mu$$

$$\leqslant \mu(R(n))\, 4M^2.$$

Since $\mu(R(n)) \to 0$, $\|A_n^\lambda f_n - B_n^\lambda f_n\| \to 0$. Using this in (6) along with the fact $A_n^\lambda \Rightarrow P^\lambda$, it follows $B_n^\lambda f \to P^\lambda f$ for f bounded. Proceed by approximation to the general case and the theorem is proved.

COROLLARY 3.2. *If T admits an approximation by partitions with speed θ/n and $\theta < 2$ with respect to $\{\xi(n)\}$, and $T_{\xi(n)}$ is periodic with period $h(n)$, then*

(i) *$P_n^\lambda \Rightarrow P^\lambda$ for each $\lambda \in$ lim inf $\Lambda(T_{\xi(n)})$ and, in particular, $P_n^1 \Rightarrow P^1$*
and

(ii) *$P_n^\lambda \Rightarrow P^\lambda$ for all $\lambda \in K$ if and only if $\Lambda(T) \subset$ lim inf $\Lambda(T_{\xi(n)})$, where P_n^λ is the projection on $H_n^\lambda = \{g \in L_2(X) \colon U_n g = \lambda g\}$ with $U_n g = g(T_{\xi(n)})$ and* lim inf $\Lambda(T_{\xi(n)}) = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty \Lambda(T_{\xi(n)})$.

Proof. Fix $\lambda \in \liminf \Lambda(T_{\xi(n)})$. Then there exists an N such that $\lambda \in \Lambda(T_{\xi(n)})$ for all $n \geqslant N$. Since $T_{\xi(n)}$ has period $h(n)$, by Corollary 3.1.2, $\Lambda(T_{\xi(n)}) = \Lambda(h(n))$, where $\Lambda(h(n))$ denotes the $h(n)$ roots of unity. Thus, $\lambda^{h(n)} = 1$ for all $n \geqslant N$. Hence, using the same corollary, $P_n^\lambda = (1/h(n)) \sum_{j=0}^{h(n)-1} \lambda^{-j} U_n^j$ for $n \geqslant N$. By the theorem, $P_n^\lambda \Rightarrow P^\lambda$. By noting that 1 is always contained in $\liminf \Lambda(T_{\xi(n)})$, the first assertion follows.

To establish (ii), first assume $\Lambda(T) \subset \liminf \Lambda(T_{\xi(n)})$. If $\lambda \in \Lambda(T)$, then by (i), $P_n^\lambda \Rightarrow P^\lambda$. So let $\lambda \in K - \Lambda(T)$. Then $P^\lambda f = 0$ for all $f \in H$. Since $P_n^\lambda = B_n^\lambda = \sum_{j=0}^{h(n)-1} \lambda^{-j} U_n^j$ or $P_n^\lambda = 0$, then in either case, $\|P_n^\lambda f\| \leqslant \|B_n^\lambda f\|$. By the theorem, $\|B_n^\lambda f\| \to \|P^\lambda f\| = 0$, and so $\|P_n^\lambda f\| \to 0$. Thus $P_n^\lambda f \to P^\lambda f$.

To show the converse, proceed by contradiction and assume $P_n^\lambda \Rightarrow P^\lambda$ for all $\lambda \in K$ and $\Lambda(T) \not\subset \liminf \Lambda(T_{\xi(n)})$. Hence there eixsts a $\lambda \in \Lambda(T)$ and a subsequence $\{n(j)\}$ such that $\lambda^{h(n(j))} \neq 1$ for $j = 1, 2, \ldots$. Since $\lambda \in \Lambda(T)$, there exists an $f \in H^\lambda$ with $f \neq 0$ and $P^\lambda f = f$. By our assumption, $P_{n(j)}^\lambda f \to P^\lambda f = f$ as $j \to \infty$. Since $\lambda^{h(n(j))} \neq 1$, $P_{n(j)}^\lambda = 0$ for all j. By the above convergence, we produce the contradiction that $f = 0$.

Remark 1. If T admits a cyclic a.p.t. with speed $f(n)$ with respect to $\{\xi(n)\}$ and $\{T_{\xi(n)}\}$ with $T_{\xi(n)}$ arbitrary, then by possibly redefining $T_{\xi(n)}$ on $C_n(h(n))$ by $T_{\xi(n)}(x) = T_{\xi(n)}^{-h(n)}(x)$ and on $R(n)$ as the identity function, $T_{\xi(n)}$ is now periodic with period $h(n)$ which is the smallest possible period. The speed function $f(n)$ does not change since (ii) in Definition 2.4 is independent of the particular $T_{\xi(n)}$ chosen.

Remark 2. For an arbitrary $T_{\xi(n)}$, $\Lambda(T_{\xi(n)}) \supset \Lambda(h(n))$. This follows by observing for $\lambda \in \Lambda(h(n))$, if f_n^λ is defined by $f_n^\lambda(x) = \lambda^{j-1}$ on $C_n(j)$ for $j = 1, 2, \ldots, h(n)$ and $f_n^\lambda(x) = 0$ on $R(n)$, then $f_n^\lambda(T_{\xi(n)}(x)) = \lambda f_n^\lambda(x)$ for $x \in X$ and so $\lambda \in \Lambda(T_{\xi(n)})$. Thus if $T_{\xi(n)}$ is chosen to be periodic with period $h(n)$ for each n, then $\liminf \Lambda(T_{\xi(n)})$ is minimal. This is desirable because of the set containment developed in the previous corollary.

4. Approximation and Projections

In light of the remarks at the end of the previous section, in this section if T admits a cyclic a.p.t. with respect to $\{\xi(n)\}$, we will assume the corresponding approximating transformations $\{T_{\xi(n)}\}$ are periodic with period $h(n)$.

Our objective is to investigate conditions for strong convergence of $\{P_n^\lambda\}$ to P^λ as mentioned in Corollary 3.2 and to apply this theory to strengthen several previously established results.

The conclusions of the following two theorems along with their first corollaries were proven in [7] for a transformation T produced by a stacking procedure. Such a T represents a specific example of an approximation by partitions. However, in light of what has been done here, the methods of proof can be generalized with slight modification to the general case of cyclic a.p.t. with speed θ/n and $\theta < 2$. We give the proof of Theorem 4.1 and Corollary 4.1.1 to illustrate some of the ideas used in [7] and the modifications required here.

THEOREM 4.1. *If T admits a cyclic a.p.t. with speed θ/n and $\theta < 2$, then $\Lambda(T) \cap K' \subset \liminf \Lambda(T_{\xi(n)})$, where K' is the subset of K consisting of all roots of unity.*

Proof. Let $\lambda \in \Lambda(T) \cap K'$. We may assume $\lambda \neq 1$. Now $\lambda \in K'$ implies $\lambda^k = 1$ for some positive integer k and assume k is the least such positive integer. Let $h(n) = a(n)\,k + r(n)$, where $0 \leqslant r(n) < k$. We have $|\lambda^{h(n)} - 1| = |\lambda^{a(n)k + r(n)} - 1| = |\lambda^{r(n)} - 1| \to 0$ as $n \to \infty$ by Lemma 2.1.

Assume there exists a subsequence $\{r(n_j)\}$ such that $r(n_j) > 0$ for all j. For any exponent $r(n_j)$ with $0 < r(n_j) < k$, $|\lambda^{r(n_j)} - 1| \geqslant |e^{2\pi i/k} - 1| \neq 0$, which contradicts the convergence of this subsequence to 1. Therefore no such subsequence exists. This implies $r(n) = 0$ for n sufficiently large, say, $n \geqslant N$. Thus $\lambda^{h(n)} = \lambda^{ka(n)} = 1$ for all $n \geqslant N$. Hence $\lambda \in \liminf \Lambda(T_{\xi(n)})$ and the conclusion follows.

Intuitively speaking, this result indicates that if λ is an eigenvalue of T and a root of unity, then it must be an eigenvalue of all the $T_{\xi(n)}$'s from some point on.

COROLLARY 4.1.1. *If T admits a cyclic a.p.t. with speed $f(n)=\theta/n$ and $\theta<2$, and there exists a subsequence $\{h(n_m)\}$ such that $h(n_{2m})$ and $h(n_{2m-1})$ are relatively prime for $m=1,2,\ldots$, then T is completely ergodic.*

Proof. Choose $\lambda\in K'$. If a and b are two integers with $\lambda^a=\lambda^b=1$, then it follows $\lambda^g=1$, where $g=(a,b)$. If $\lambda\in\liminf\Lambda(h(n))$, then there is an N such that $\lambda^{h(n)}=1$ for all $n\geqslant N$. Choose m so large such that $n_{2m-1}\geqslant N$. Then $\lambda^{h(n_{2m})}=\lambda^{h(n_{2m-1})}=1$. Since $(h(n_{2m}),h(n_{2m-1}))=g=1$, $\lambda^g=\lambda=1$ and so $\liminf\Lambda(h(n))=\{1\}$. By the theorem, $\Lambda(T)\cap K'=\{1\}$.

If T is not completely ergodic, there exists a $\lambda\in\Lambda(T)$ with $\lambda\neq 1$ and $\lambda^k=1$ for some integer k. By the above, this is impossible and so the corollary in proved.

Using Theorem 4.1 and Corollary 3.2, the following is established.

COROLLARY 4.1.2. *If T admits an approximation by partitions with speed θ/n and $\theta<2$, then $P_n^\lambda\Rightarrow P^\lambda$ for all $\lambda\in\Lambda(T)\cap K'$.*

The proofs of the following theorem and its corollary are almost identical to those of Theorem 2 and its ensuing corollary in [7] and so are omitted here.

THEOREM 4.2. *If T admits a cyclic a.p.t. in (k,j) pairs with speed θ/n and $\theta<2$, then $\Lambda(T)\subset\liminf\Lambda(T_{\xi(n)})$.*

COROLLARY 4.2.1. *If T admits a cyclic a.p.t. in (k,j) pairs with speed θ/n and $\theta<2$, then*

(i) *T is weakly mixing if and only if T is completely ergodic, and*

(ii) *if there exists a subsequence $\{h(n_m)\}$ with $(h(n_{2m}),h(n_{2m-1}))=1$ for $m=1,2,\ldots$ or, in particular, if the (k,j) pair is a k-pair subsequence, then T is weakly mixing.*

We remark that the subsequence $\{h(n_m)\}$ in (ii) does no have to be the same subsequence which is the (k,j) pair subsequence. However, if T admits a cyclic a.p.t. in k-pairs, which is a (k,j) pair with $j=1$, then the corresponding subsequence $h(n_m)$ where $h(n_{2m})=kh(n_{2m-1})+1$ for $m=1,2,\ldots$ clearly satisfies $(h(n_{2m}),h(n_{2m-1}))=1$ for each m. Thus (ii) of this corollary is a generalization of Theorem 2.4.

COROLLARY 4.2.2. *If T admits an approximation by partitions in (k,j) pairs with speed θ/n and $\theta<2$, then*

(i) *$P_n^\lambda\Rightarrow P^\lambda$ for all $\lambda\in K$ and*

(ii) *the following are equivalent:*

(a) T *is weakly mixing,*

(b) T *is completely ergodic,*

(c) $P_n^\lambda \Rightarrow 0$ *for all* $\lambda \in K$, $\lambda \neq 1$.

Proof. Statement (i) is a consequence of the theorem along with Corollary 3.2. In statement (ii), (a) and (b) are equivalent by Corollary 4.2.1. If T is weakly mixing, then $\Lambda(T) = \{1\}$. Thus $P^\lambda = 0$ for $\lambda \neq 1$ and, along with (i), $P_n^\lambda \Rightarrow 0$ for all $\lambda \in K$, $\lambda \neq 1$. The converse follows similarly.

THEOREM 4.3. *If* T *admits a cyclic a.p.t. with respect to* $\{\xi(n)\}$ *with arbitrary speed, then* T *has discrete spectrum with* $\Lambda(T) = \liminf \Lambda(T_{\xi(n)})$ *and* $P_n^\lambda \Rightarrow P^\lambda$ *for all* $\lambda \in K$.

Proof. By Theorem 2.1, T has discrete spectrum and $\Lambda(T) \subset K'$. Employing Theorem 4.1, $\Lambda(T) \subset \liminf \Lambda(T_{\xi(n)})$.

Now, by an appropriate choice of a speed function $f(n)$, $\sum_{j=1}^{h(n)} \mu(TC_n(j)\,\Delta T_{\xi(n)} C_n(j)) = 0$ and so T admits an approximation by partitions with arbitrary speed. For $\lambda \in \liminf \Lambda(T_{\xi(n)})$ and for each n, define f_n^λ as in Remark 2. That is, $f_n^\lambda(x) = \lambda^{j-1}$ for $x \in C_n(j)$, $j = 1, 2, \ldots, h(n)$. Choose N large enough so that $\lambda^{h(n)} = 1$ for $n \geqslant N$. It is easy to see $f_n^\lambda(T(x)) = \lambda f_n^\lambda(x)$ for $n \geqslant N$ and so $\lambda \in \Lambda(T)$. The rest of the theorem follows using Corollary 3.2.

One may ask if strong convergence of $\{P_n^\lambda\}$ to P^λ always occurs. In [6], a transformation T was produced by a stacking method with the required speed with a non-root of unity λ as an eigenvalue. Since λ is a non-root of unity $P_n^\lambda = 0$ for all n and so the following theorem results.

THEOREM 4.4. *There exists a transformation* T *which admits an approximation by partitions with speed* θ/n *and* $\theta < 2$ *and a* $\lambda \in \Lambda(T)$ *such that* $\{P_n^\lambda\}$ *does not converge strongly to* P^λ.

REFERENCES

1. M. A. AKCOGLU, R. V. CHACON, AND T. SCHWARTZBAUER, Commuting transformations and mixing, *Proc. Amer. Math. Soc.* **24** (1970), 637–642.
2. R. V. CHACON, Approximation of transformations with continuous spectrum, *Pacific J. Math.* **31** (1969), 293–302.
3. P. R. HALMOS, "Lectures on Ergodic Theory," Math. Soc. of Japan, Chelsea, New York, 1956.
4. A. B. KATOK AND A. M. STEPIN, Approximation of ergodic dynamic systems by periodic transformations (Russian), *Dokl. Akad. Nauk SSSR* **171** (1966), 1268–1271. [Soviet Math. Dokl. 7 (1966), 1638–1641]

5. A. B. KATOK AND A. M. STEPIN, Approximations in ergodic theory (Russian), *Uspekhi Math. Nauk* **22** (5) (1967), 81–106. [Russian Math. Surveys **22** (5) (1967), 77–102]
6. D. QUIRING AND R. J. TUREK, A stacking method in ergodic theory, *Indiana Univ. Math. J.* **25** (1976), 493–498.
7. R. J. TUREK, Eigenvalues and stacking methods in ergodic theory, *Indiana Univ. Math. J.* **27** (1978), 1–8.